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Self-Injective Group Algebras

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For some time now it has been known that the group algebra of a finite group is self-injective [1]. The converse of this result was proved with some difficulty by Renault and others [3]. The purpose of this note is to give a short proof of this theorem.

G will denote a group (possibly infinite), K will denote a field, and $K[G]$ will be the discrete group algebra of G over K. We show that if $K[G]$ is self-injective as a right module then G is finite.

We begin with the

LEMMA. *If H is a subgroup of G and $K[G]$ is self-injective then $K[H]$ is self-injective.*

Proof. Let $g_1 = 1, g_2, \ldots$ be a set of coset representatives for H in G and let θ be the vector space projection of $K[G]$ onto $K[H]$. It suffices to show that for any right ideal I of $K[H]$ the following diagram can be completed:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & K[H] \arrow[dl, dashed, "\phi"] \\ & K[H] & \end{tikzcd}$$

$J = \sum_i Ig_i$ is a right ideal of $K[G]$. Define $\hat{f} : J \to K[G]$ by $\hat{f}(\sum a_i g_i) = \sum f(a_i)\, g_i$ where $a_i \in I$. If $u \in G$ then $g_i u = h_{ij} g_j$ for some $h_{ij} \in H$.

$$\hat{f}\left(\left(\sum a_i g_i\right) u\right) = \hat{f}\left(\sum_i a_i h_{ij} g_j\right) = \sum f(a_i h_{ij}) g_j = \sum f(a_i) h_{ij} g_j$$
$$= \sum f(a_i) g_i u = \hat{f}\left(\sum a_i g_i\right) \cdot u.$$

Thus $\hat{f}$ is a $K[G]$-map.

By self-injectivity $\exists \hat{\phi} : K[G] \to K[G] \ni \hat{\phi}\,|_J = \hat{f}$. Set $\phi = \theta(\hat{\phi}\,|_{K[H]})$. Then

$\phi|_I = \theta(\hat{\phi}|_I) = \theta(\hat{f}|_I) = \theta f = f$. ϕ is a $K[H]$-map: if $h \in H$, $\alpha \in K[H]$ then

$$\phi(\alpha h) = \theta(\hat{\phi}(\alpha h)) = \theta(\hat{\phi}(\alpha)h) = \theta(\hat{\phi}(\alpha)) \cdot h = \phi(\alpha) \cdot h.$$

LEMMA. *If $K[G]$ is self-injective then G is locally finite.*

Proof. The following result is known [2]: if R is a self-injective ring then every proper finitely generated left ideal has a proper right annihilator. Let $g_1, g_2, ..., g_n \in G$. $K[G](1 - g_1) + K[G](1 - g_2) + \cdots + K[G](1 - g_n)$ is a left ideal of $K[G]$ in the augmentation ideal which is proper. Thus there is a non-zero $a \in K[G] \ni (1 - g_i)a = 0$ for $i = 1, ..., n$. This implies that the group generated by $\{g_1, g_2, ..., g_n\}$ is finite [1].

THEOREM. *If $K[G]$ is self-injective then G is finite.*

Proof. Suppose $K[G]$ is self-injective and G is infinite. By the two lemmas we may assume that $G = \bigcup_{i=1}^{\infty} G_i$ where $G_1 \subsetneqq G_2 \subsetneqq G_3 \subsetneqq \cdots$ is a strictly increasing chain of finite subgroups. Set $|G_n| = S_n$.

Inductively define $a_n \in K[G_n]$ by $a_1 = 1$ and $a_{n+1} = a_n + \sum_{g \in G_n} g$. Then

$$a_{n+1} = \Big(\sum_{g \in G_n - G_{n-1}} g\Big) + 2\Big(\sum_{g \in G_{n-1} - G_{n-2}} g\Big) + \cdots + (n + 1).$$

If I is the augmentation ideal of $K[G]$ define $f : I \to K[G]$ as follows: if $g \in G_n$ set $f(g - 1) = a_n \cdot (g - 1)$. We show f is well-defined by showing that $g' \in G_n \subseteq G_{n+1}$ implies $(a_{n+1} - a_n)(g' - 1) = 0$. But

$$(a_{n+1} - a_n)(g' - 1) = \Big(\sum_{g \in G_n} g\Big)(g' - 1) = 0.$$

It is now immediate that f is a right $K[G]$-map.

If $K[G]$ is self-injective $\exists p \in K[G] \ni f(g - 1) = p \cdot (g - 1)\ \forall g \in G$. Let p_n be the image of p under the vector space projection $K[G] \to K[G_n]$.

$$\forall g \in G_n,\ p_n \cdot (g - 1) = a_n \cdot (g - 1) \Rightarrow (p_n - a_n) \cdot (g - 1) = 0$$

$$\Rightarrow \exists k \in K \ni p_n = a_n + k \cdot \Big(\sum_{g \in G_n} g\Big).$$

If $k = 0$ then the support of p_n contains $G_{n-1} - G_{n-2}$ which implies $|\operatorname{Supp} p_n| \geqslant S_{n-1} - S_{n-2} \geqslant S_{n-2}$. If $k \neq 0$ then the support of p_n contains a coset of G_{n-1} other than G_{n-1} itself; this implies $|\operatorname{Supp} p_n| \geqslant S_{n-1}$. In any event, $|\operatorname{Supp} p_n| \geqslant S_{n-2}$ and $S_{n-2} \to \infty$. Thus p cannot have finite support, a contradiction.

References

1. I. G. Connell, On the group ring, *Canad. J. of Math.* **15** (1963), 650–685.
2. D. S. Passman, On the ring of quotients of a group ring, to appear.
3. G. Renault, Sur les anneaux de groupes, *Compt. Rend. Ser. A* **273** No. 2 (1971), 84–88.